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B.Sc- part-II(H), Paper-III
Real Analysis Model Question.
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or $a_n^{\frac{1}{n}} = \left[\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$

or, $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right) - \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right]^{-1}$

$= [e-1]^{-1} = \frac{1}{e-1} < 1$, as $2<e<3$.

Hence, by Cauchy's nth root test, the given series is convergent.

Theorem: 06. State and prove Leibnitz's Test:

Soln: Statement: →

$\sum_{1}^{\infty} (-1)^{n-1} a_n$ i.e. $a_1 - a_2 + a_3 - a_4 + \cdots$ to ∞

is convergent if

(I) each term is numerically less than the preceding i.e.

$|a_{n+1}| < |a_n|$, for all n, and

II $a_n \to 0$ as $n \to \infty$.

Proof: → Let $S_{2n} = (a_1-a_2) + (a_3-a_4) + \cdots + (a_{2n-1} - a_{2n})$. —— (A)

and this can be written in the form

$S_{2n} = a_1 - (a_2-a_3) - (a_4-a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$ —— (B)

Since $a_1 > a_2 > a_3 > \cdots$, therefore

$a_1-a_2,\ a_3-a_4, \cdots$ all are positive.

Also, $a_2-a_3, a_4-a_5, \cdots$ all are positive.

From (A), it, therefore, follows that S_{2n} is positive and increases with n, and from (B) it is seen that S_{2n} is always less than the fixed number a_1.

Hence, S_{2n} tends to a limit which is less than a_1.

Again, $S_{2n+1} = S_{2n} + a_{2n+1}$.

$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$.

Since, $\lim_{n\to\infty} a_{2n+1} = 0$,

$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}$.

The series is therefore convergent, its sum is positive and less than a_1.

1. Test the convergence of the series (29)

$$\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \frac{1}{x+3a} + \cdots \text{ to } \infty.$$

x and a being positive.

Soln: We have $x>0$ and $a>0$. The terms of the given series are alternately positive and negative.

Again, $\frac{1}{x} > \frac{1}{x+a} > \frac{1}{x+2a} > \frac{1}{x+3a} > \cdots\cdots$

That is $u_n > u_{n+1}$, for all $n \in N$, (numerically).

Therefore each term of the given series is numerically less than the preceding term.

Again, numerically the nth term, $u_n = \frac{1}{x+(n-1)a}$

or $\lim_{n \to \infty} u_n = 0$

Hence, the given series satisfies all the conditions of Leibnitz's theorem. Thus the given series is convergent.

Q.2. Examine the convergence of the series

$$\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots \text{ to } \infty.$$

Soln: We have $\log 2 > 0$, $\log 3 > 0$, $\log 4 > 0$, etc.

The terms of the given series are alternately positive and negative.

$\because \log 2 < \log 3 < \log 4 < \log 5 < \cdots\cdots$

$\therefore \frac{1}{\log 2} > \frac{1}{\log 3} > \frac{1}{\log 4} > \frac{1}{\log 5} > \cdots\cdots$

That is, $a_n > a_{n+1}$ (numerically), for all $n \in N$.

Therefore, each term of the given series is numerically less than the preceding term.

Again, a_n (numerically) $= \frac{1}{\log(n+1)}$

or, $\lim_{n \to \infty} a_n$ (numerically) $= 0$.

We find that the given series satisfies all the conditions of Leibnitz's theorem. Hence, the given series is convergent.

Theorem 7. State and prove Comparison Test.

Soln: Statement:

(I) If Σa_n and Σb_n be two positive term series and after some particular term $\frac{a_n}{a_{n-1}} < \frac{b_n}{b_{n-1}}$, then Σa_n converges provided that Σb_n converges.

II If Σa_n and Σb_n be two positive term series and after some particular term $\frac{a_n}{a_{n-1}} > \frac{b_n}{b_{n-1}}$, then Σa_n diverges provided that Σb_n diverges.

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Proof. I. Let the series beginning after some particular term be $a_1+a_2+a_3+\cdots+a_n+\cdots$ to ∞,
and $b_1+b_2+b_3+\cdots+b_n+\cdots$ to ∞.
It is given that

$$\frac{a_2}{a_1} < \frac{b_2}{b_1}$$

$$\frac{a_3}{a_4} < \frac{b_3}{b_2}$$

$$\cdots\cdots$$

$$\frac{a_n}{a_{n-1}} < \frac{b_n}{b_{n-1}}$$

Multiplying all these inequalities, we get

$$\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdots\frac{a_n}{a_{n-1}} < \frac{b_2}{b_1}\cdot\frac{b_3}{b_2}\cdots\frac{b_n}{b_{n-1}}$$

or, $\frac{a_n}{a_1} < \frac{b_n}{b_1}$ or $a_n < \left(\frac{a_1}{b_1}\right) b_n$.

Let $\frac{a_1}{b_1} = K$. Then $a_n < K b_n$. or $\Sigma a_n < K \Sigma b_n$.

If Σb_n converges, then Σa_n also converges.

Proof II. Let the series beginning after some particular term be
$a_1+a_2+a_3+\cdots+a_n+\cdots$ to ∞.
and $b_1+b_2+b_3+\cdots+b_n+\cdots$ to ∞.
It is given that $\frac{a_2}{a_1} > \frac{b_2}{b_1}$

$$\frac{a_3}{a_2} > \frac{b_3}{b_2}$$

$$\cdots\cdots$$

$$\frac{a_n}{a_{n-1}} > \frac{b_n}{b_{n-1}}$$

Multiplying all these inequalities, we get

$$\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdots\frac{a_n}{a_{n-1}} > \frac{b_2}{b_1}\cdot\frac{b_3}{b_2}\cdots\frac{b_n}{b_{n-1}}$$

or, $\frac{a_n}{a_1} > \frac{b_n}{b_1}$ or $a_n > \left(\frac{a_1}{b_1}\right) b_n$

Let $\frac{a_1}{b_1} = K$.

Then, $a_n > K b_n$ or $\Sigma a_n > K \Sigma b_n$.

If Σb_n is divergent, then Σa_n is also divergent.

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Soln: Statement: The positive term series Σa_n converges or diverges according as $\lim\limits_{n\to\infty} \left\{ n\left(\frac{a_n}{a_{n+1}} - 1\right) \right\} >$ or < 1

Proof: Let us compare the given series Σa_n with the auxiliary series Σb_n.

Where $\Sigma b_n = \Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \frac{1}{(n+1)^p} + \cdots$ to ∞

We know that Σb_n is convergent or divergent according according as $p >$ or < 1.

Hence, by comparision test, the given series Σa_n will be convergent or divergent according as

$$\frac{a_n}{a_{n+1}} > \text{ or } < \frac{b_n}{b_{n+1}}$$

$$\text{or, } \frac{a_n}{a_{n+1}} > \text{ or } < \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

$$\text{or, } \frac{a_n}{a_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)^p$$

$$\text{or, } \frac{a_n}{a_{n+1}} > \text{ or } < 1 + p\cdot\frac{1}{n} + \frac{p(p-1)}{\lfloor 2} \frac{1}{n^2} + \cdots$$

by Binomial theorem

$$\text{or, } \frac{a_n}{a_{n+1}} - 1 > \text{ or } < \frac{1}{n}\left[p + \frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n} + \cdots\right]$$

$$\text{or, } n\left(\frac{a_n}{a_{n+1}} - 1\right) > \text{ or } < p + \frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n} + \cdots$$

proceeding to limit as $n\to\infty$, we get

$$\lim_{n\to\infty} \left\{ n\left(\frac{a_n}{a_{n+1}} - 1\right) \right\} > \text{ or } < p$$

$$\text{or, } \lim_{n\to\infty} \left\{ n\left(\frac{a_n}{a_{n+1}} - 1\right) \right\} > \text{ or } < 1.$$

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Q.1. Test the convergence of the series

$$\frac{x}{2} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^3}{6} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{x^5}{10} + \cdots \text{ to } \infty.$$

Soln: Let the nth term of the given series be denoted by a_n.

Then, $a_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{7}{8}\cdots\frac{4n-7}{4n-6}\cdot\frac{4n-5}{4n-4}\cdot\frac{x^{2n-1}}{4n-2}$.

Replacing n by $(n+1)$, we get

$$a_{n+1} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{4n-5}{4n-4}\cdot\frac{4n-3}{4n-2}\cdot\frac{4n-1}{4n}\cdot\frac{x^{2n+1}}{4n+2}.$$

Then, $\frac{a_{n+1}}{a_n} = \frac{4n-3}{4n-2}\cdot\frac{4n-1}{4n}\cdot\frac{4n-2}{4n+2}x^2 = \frac{4-\frac{3}{n}}{4}\cdot\frac{4-\frac{1}{n}}{4+\frac{2}{n}}\cdot x^2$